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Theta functions and symplectic groups, II

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ABSTRACT

The main result is an application of A. Weil's method of theta functions (Coll. Papers, vol. III, 1–69) and states that, for any locally compact commutative group G isomorphic to its dual, there is an *injective* representation of the symplectic group $Sp(G)$ by a certain group of unitary operators in $L^2(G\times G)$ containing the Fourier transformation (in a slightly modified form).

Here another theme from Weil's paper 1964b in [3] will be discussed to which we also refer for terminology and notation. Let G be a locally compact commutative group, written additively. For the symplectic group $Sp(G)$ and the groups $B_0(G)$ and $\mathbf{B}_0(G)$ the following is shown in 1964b: there is a monomorphism of $Sp(G)$ into $B_0(G)$ if $x\mapsto 2x$ is an automorphism of G; also, if G is isomorphic to its dual group G^*, then a certain important part of $B_0(G)$ can be lifted to $\mathbf{B}_0(G)$ (cf. 1964b, n$^{\text{os}}$ 5, 13; [3], pp. 8–9, 18). Here the groups $B_0(G^2)$ and $\mathbf{B}_0(G^2)$, where $G^2:=G\times G$, will be considered instead: it will be shown that there is always a monomorphism of $Sp(G)$ into $B_0(G^2)$ such that the image of $Sp(G)$ can be lifted to $\mathbf{B}_0(G^2)$, i.e. *$Sp(G)$ can be isomorphically embedded in* $\mathbf{B}_0(G^2)$; moreover, the embedding is a natural one, in a sense to be discussed at the end of the paper. This result is analogous to that proved for $B_0(G)$ in [2], part III, and Weil's method of theta functions is used again[1].

[1] A preliminary version was communicated at the meeting on Harmonic Analysis held in September 1980 at the Universität Paderborn.

Let the duality between G^2 and G^{*2} be given as in [2], (III–3). Let

$$\mathfrak{z}_j=(x_j,y_j,x_j^*,y_j^*)\in G^2\times G^{*2},\quad j=1,2,$$

and put

$$(1)\qquad F_1(\mathfrak{z}_1,\mathfrak{z}_2):=\langle x_1,x_2^*\rangle,\quad F_2(\mathfrak{z}_1,\mathfrak{z}_2):=\langle y_1,y_2^*\rangle.$$

The function $F^{[2]}$ introduced in [2], (III–4) can now be written as the product

$$(2)\qquad F^{[2]}(\mathfrak{z}_1,\mathfrak{z}_2)=F_1(\mathfrak{z}_1,\mathfrak{z}_2)\cdot F_2(\mathfrak{z}_1,\mathfrak{z}_2).$$

Consider an element

$$\sigma=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in Sp(G)$$

and write the defining property of the symplectic group (1964b,n° 2; [3], p. 6) as follows:

$$\frac{\langle x_1\alpha+x_1^*\gamma,x_2\beta+x_2^*\delta\rangle}{\langle x_1,x_2^*\rangle}=\frac{\langle x_2\alpha+x_2^*\gamma,x_1\beta+x_1^*\delta\rangle}{\langle x_2,x_1^*\rangle}.$$

It will be useful to change the notation,

$$x_1\mapsto x,\quad x_2^*\mapsto x^*,\quad x_2\mapsto y,\quad x_1^*\mapsto y^*,$$

so that the symplectic property assumes the form:

$$(3)\qquad \frac{\langle x\alpha+y^*\gamma,y\beta+x^*\delta\rangle}{\langle x,x^*\rangle}=\frac{\langle y\alpha+x^*\gamma,x\beta+y^*\delta\rangle}{\langle y,y^*\rangle}$$

for arbitrary x,y in G, x^*,y^* in G^*.

Define next an automorphism $\sigma^{(2)}$ of $G^2\times G^{*2}$ by

$$(4)\qquad \sigma^{(2)}:=\begin{pmatrix}\alpha&0&0&\beta\\0&\alpha&\beta&0\\0&\gamma&\delta&0\\\gamma&0&0&\delta\end{pmatrix},$$

where 0 denotes a zero morphism. Put, with F_1 as in (1),

$$(5)\qquad \psi_\sigma^{(2)}(\mathfrak{z}):=F_1(\mathfrak{z}\sigma^{(2)},\mathfrak{z}\sigma^{(2)})/F_1(\mathfrak{z},\mathfrak{z}),\quad \mathfrak{z}:=(x,y,x^*,y^*)\in G^2\times G^{*2}.$$

$\psi_\sigma^{(2)}$ is precisely the term on the left in (3). Let us show: $(\sigma^{(2)},\psi_\sigma^{(2)})$ *belongs to* $B_0(G^2)$, i.e. for $\mathfrak{z}_1,\mathfrak{z}_2$ in $G^2\times G^{*2}$ the relation

$$(6)\qquad \psi_\sigma^{(2)}(\mathfrak{z}_1+\mathfrak{z}_2)=\psi_\sigma^{(2)}(\mathfrak{z}_1)\cdot\psi_\sigma^{(2)}(\mathfrak{z}_2)\cdot F^{[2]}(\mathfrak{z}_1\sigma^{(2)},\mathfrak{z}_2\sigma^{(2)})/F^{[2]}(\mathfrak{z}_1,\mathfrak{z}_2)$$

holds; in particular, $\sigma^{(2)}$ is in $Sp(G^2)$ and $\psi_\sigma^{(2)}$ is a character of the second degree on $G^2\times G^{*2}$.

To prove (6), let $\mathfrak{z}:=\mathfrak{z}_1+\mathfrak{z}_2$ in (5). Since

$$F_1(\mathfrak{z}_1+\mathfrak{z}_2,\mathfrak{z}_1+\mathfrak{z}_2)=F_1(\mathfrak{z}_1,\mathfrak{z}_1)\cdot F_1(\mathfrak{z}_2,\mathfrak{z}_2)\cdot F_1(\mathfrak{z}_1,\mathfrak{z}_2)\cdot F_1(\mathfrak{z}_2,\mathfrak{z}_1),$$

we have

$$
(7) \qquad \begin{cases} \psi_\sigma^{(2)}(\mathfrak{z}_1+\mathfrak{z}_2)/[\psi_\sigma^{(2)}(\mathfrak{z}_1)\cdot\psi_\sigma^{(2)}(\mathfrak{z}_2)] = \\ = [F_1(\mathfrak{z}_1\sigma^{(2)},\mathfrak{z}_2\sigma^{(2)})/F_1(\mathfrak{z}_1,\mathfrak{z}_2)]\cdot[F_1(\mathfrak{z}_2\sigma^{(2)},\mathfrak{z}_1\sigma^{(2)})/F_1(\mathfrak{z}_2,\mathfrak{z}_1)]. \end{cases}
$$

Here we apply (3): putting there $x=x_2$, $y=y_1$, $x^*=x_1^*$, $y^*=y_2^*$, we obtain by the very definitions (1), (4)

$$
F_1(\mathfrak{z}_2\sigma^{(2)},\mathfrak{z}_1\sigma^{(2)})/F_1(\mathfrak{z}_2,\mathfrak{z}_1)=F_2(\mathfrak{z}_1\sigma^{(2)},\mathfrak{z}_2\sigma^{(2)})/F_2(\mathfrak{z}_1,\mathfrak{z}_2),
$$

and substituting this in (7) we get (6) [cf. (2)].

From the law of multiplication in $B_0(G^2)$ (cf. 1964b, n° 5, formula (7); [3], p. 8) and the definitions (4), (5) of $\sigma^{(2)}$ and $\psi_\sigma^{(2)}$ it readily follows that $\sigma\mapsto(\sigma^{(2)},\psi_\sigma^{(2)})$ *is a monomorphism of* $Sp(G)$ *into* $B_0(G^2)$. Let $Sp'(G^2)$ be the image of $Sp(G)$ in $B_0(G^2)$ under this monomorphism; we want to show that $Sp'(G^2)$ *can be lifted to* $\mathbf{B}_0(G^2)$, by a suitable application of Weil's theta method.

For this purpose we introduce the subgroup

$$
(8a) \qquad \Gamma := \{(t,0)\,|\,t\in G\}
$$

of G^2. The associated subgroup Γ_* in G^{*2} is

$$
(8b) \qquad \Gamma_* = \{(0,t^*)\,|\,t^*\in G^*\},
$$

so

$$
\Gamma\times\Gamma_* = \{(t,0,0,t^*)\,|\,t\in G, t^*\in G^*\}.
$$

It is readily verified that

(i) $\sigma^{(2)}$ induces an automorphism of $\Gamma\times\Gamma_*$, of Haar modulus 1,

(ii) $\psi_\sigma^{(2)}(\zeta)=1$ for all $\zeta\in\Gamma\times\Gamma_*$.

This means that $Sp'(G^2)$ belongs to $B_0(G^2,\Gamma)$, so that Weil's theta method is applicable. Let Φ be in $\mathfrak{S}^1(G^2)$ ([2], part I); the theta transform of Φ relative to the subgroup Γ of G^2 given by (8a) is

$$
(9) \qquad \Theta_\Phi(\mathfrak{z})=(\tilde{T}_\Gamma\Phi)(\mathfrak{z}) := \int_G \Phi(x+t,y)\cdot\langle t,x^*\rangle dt, \quad \mathfrak{z}=(x,y,x^*,y^*).
$$

See [2], part II for references to Weil's general theta method and part III for another example. The function (9) is, in fact, independent of y^*, in accordance with the general theory.

We now obtain an embedding of $Sp(G)$ in $\mathbf{B}_0(G^2)$,

$$
\sigma\mapsto\mathbf{r}^{(2)}(\sigma), \quad \sigma\in Sp(G), \quad \mathbf{r}^{(2)}(\sigma)\in\mathbf{B}_0(G^2),
$$

by putting

$$
(10) \qquad \mathbf{r}^{(2)}(\sigma) := \tilde{T}^{-1}\tilde{\mathbf{r}}(\sigma^{(2)},\psi_\sigma^{(2)})\tilde{T}, \quad \sigma\in Sp(G),
$$

where $\tilde{T}=\tilde{T}_\Gamma$ is defined by (9) and $\tilde{\mathbf{r}}$ is the theta representation of $Sp'(G^2)\subset \subset B_0(G^2,\Gamma)$:

$$(11)\qquad [\tilde{\mathbf{r}}(\sigma^{(2)},\psi_\sigma^{(2)})\Theta](\mathfrak{z}):=\Theta(\mathfrak{z}\sigma^{(2)})\cdot\psi_\sigma^{(2)}(\mathfrak{z}),\quad \Theta\in L^2(G^2,\Gamma),$$

$(\sigma^{(2)},\psi_\sigma^{(2)})\in Sp'(G^2)$ being as in (4), (5).

Let us show that the embedding (10) can be expressed in a rather simple way. Put, for $\mathfrak{z}=(x,y,x^*,y^*)\in G^2\times G^{*2}$, and with F_1 as in (1),

$$(12)\qquad q(\mathfrak{z}):=F_1(\mathfrak{z},\mathfrak{z})=\langle x,x^*\rangle.$$

Then we have:

$$(13)\qquad q(\mathfrak{z})\cdot(\tilde{T}\Phi)(\mathfrak{z})=\int_G \Phi(t,y)\cdot\langle t,x^*\rangle dt.$$

This means that the function $q\cdot\tilde{T}\Phi$ on $G^2\times G^{*2}$ is $\Gamma\times\Gamma_*$-periodic, and we can define a function $\mathring{T}\Phi$ on $G\times G^*$ by putting

$$(14)\qquad \mathring{T}\Phi(\pi\mathfrak{z}):=q(\mathfrak{z})\cdot(\tilde{T}\Phi)(\mathfrak{z}),$$

with

$$(15)\qquad \pi\mathfrak{z}:=(y,x^*)\in G\times G^* \text{ for } \mathfrak{z}=(x,y,x^*,y^*)\in G^2\times G^{*2},$$

so that π is a surjective morphism

$$(16)\qquad \pi: G^2\times G^{*2}\to G\times G^*$$

with kernel $\Gamma\times\Gamma_*$. The function q defined by (12) satisfies the relation (III–18) in [2], with Γ now defined by (8a) above; thus $\Theta\mapsto q\cdot\Theta$ is an isomorphic mapping of $L^2(G^2,\Gamma)$ on to $L^2(G\times G^*)$, and the figure in [2] following (III–18) also applies in the present case (with the same 'abus de notation' as explained there).

In $L^2(G\times G^*)$ there is an obvious realization of $Sp(G)$ as a group of unitary operators: we put for $\varphi\in L^2(G\times G^*)$

$$(17)\qquad [\mathbf{r}''(\sigma)\varphi](y,x^*):=\varphi((y,x^*)\sigma),\quad \sigma\in Sp(G).$$

The theta representation $\tilde{\mathbf{r}}$ of $Sp'(G^2)$ defined by (11) is related to $\mathbf{r}''$ by

$$(18)\qquad \tilde{\mathbf{r}}(\sigma^{(2)},\psi_\sigma^{(2)})\Theta=q^{-1}\cdot\mathbf{r}''(\sigma)(q\cdot\Theta),\quad \Theta\in L^2(G^2,\Gamma),$$

as is readily seen: the definitions (5), (12) say that $\psi_\sigma^{(2)}(\mathfrak{z})=q(\mathfrak{z}\sigma^{(2)})\cdot q(\mathfrak{z})^{-1}$, thus $q(\mathfrak{z})\cdot[\tilde{\mathbf{r}}(\sigma^{(2)},\psi_\sigma^{(2)})\Theta](\mathfrak{z})$ reduces to $q(\mathfrak{z}\sigma^{(2)})\cdot\Theta(\mathfrak{z}\sigma^{(2)})$, or to $\varphi(\pi(\mathfrak{z}\sigma^{(2)}))$ if we define $\varphi\in L^2(G\times G^*)$ by $\varphi(\pi\mathfrak{z})=q(\mathfrak{z})\cdot\Theta(\mathfrak{z})$ [cf. (14)–(16)]; since $\pi(\mathfrak{z}\sigma^{(2)})=(\pi\mathfrak{z})\sigma$ ($\sigma\in Sp(G)$), (18) follows. We can write (18), with the 'abus de notation' mentioned,

$$(19)\qquad \tilde{\mathbf{r}}(\sigma^{(2)},\psi_\sigma^{(2)})=q^{-1}\mathbf{r}''(\sigma)q.$$

Substituting (19) into (10) and applying (14) – which we may write $\mathring{T}=qT$ – *we obtain for* (10) *the simple form*

$$(20)\qquad \mathbf{r}^{(2)}(\sigma)=\mathring{T}^{-1}\mathbf{r}''(\sigma)\mathring{T},\quad \sigma\in Sp(G).$$

The functions $\overset{\circ}{T}$ defined by (14) for $\Phi \in \mathfrak{S}^1(G^2)$ are given explicitly by [cf. (13)]

$$\overset{\circ}{T}\Phi(y,x^*) = \int_G \Phi(t,y)\cdot\langle t,x^*\rangle dt, \quad (y,x^*)\in G\times G^*, \tag{21}$$

and belong to $\mathfrak{S}^1(G\times G^*)$ by properties I_1 and I_4 in [2], part I; the mapping $\overset{\circ}{T}$ is here [the extension to $L^2(G^2)$ of] the 'partial' Fourier transformation (1964b, n° 11; [3], p. 17) on $\mathfrak{S}^1(G^2)$ [which is dense in $L^2(G^2)$]. The inverse transformation $\overset{\circ}{T}^{-1}$ is then, according to (14) and the general inversion formula [2], (II-4),

$$(\overset{\circ}{T}^{-1}\varphi)(x,y) = \int_{G^*} \varphi(y,t^*)\cdot\overline{\langle x,t^*\rangle}dt^*, \quad \varphi\in\mathfrak{S}^1(G\times G^*), \tag{22}$$

dt^* being the Haar measure on G^* dual to that given on G. Thus $\overset{\circ}{T}^{-1}$ is [the extension to $L^2(G\times G^*)$ of] the partial inverse Fourier transformation on $\mathfrak{S}^1(G\times G^*)$. It is easy, of course, to deduce (22) directly from (21).

We refer again to [2], part III: a comparison with the case treated there offers some interest; cf. especially (III-21)-(III-23).

The formulae (21), (17) and (22), combined with (20), show: the closed linear subspace of $L^2(G^2)$ consisting of all even functions, i.e. those $f\in L^2(G^2)$ coinciding almost everywhere with some f' such that $f'(-x,-y)=f'(x,y)$ for all $(x,y)\in G^2$, – this is a proper subspace of $L^2(G^2)$ unless all elements of G are of order 2 – is invariant under the operators $\mathbf{r}^{(2)}(\sigma)$, $\sigma\in Sp(G)$.

Using the same formulae and considering functions $\Phi\in\mathfrak{S}^1(G^2)$, we can readily determine the operator $\mathbf{r}^{(2)}(\sigma)$ explicitly for certain elements $\sigma\in Sp(G)$; we list the results here, valid for general $\Phi\in L^2(G^2)$:

$$[\mathbf{r}^{(2)}(\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{*-1}\end{pmatrix})\Phi](x,y) = |\alpha|\cdot\Phi(x\alpha,y\alpha), \tag{23a}$$

where $\alpha\in\mathrm{Aut}(G)$ and $|\alpha|$ is the Haar modulus of α;

$$[\mathbf{r}^{(2)}(\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix})\Phi](x,y) = \langle x,y\beta\rangle\cdot\Phi(x,y), \tag{23b}$$

where β is a *symmetric* morphism of G into G^*, i.e. $\langle x,y\beta\rangle=\langle y,x\beta\rangle$ – this is a character of the second degree on G^2. Finally, if there exists an isomorphism γ of G^* on to G, then

$$[\mathbf{r}^{(2)}(\begin{pmatrix}0 & -\gamma^{*-1}\\ \gamma & 0\end{pmatrix})\Phi](x,y) = |\gamma|^{-1}\cdot\Phi^*(-y\gamma^{*-1},-x\gamma^{*-1}), \tag{23c}$$

where $|\gamma|dx^*=dx$ $(x=x^*\gamma)$, dx^* being dual to dx – note that $|\gamma|^{-1}=|\gamma^{*-1}|$ (cf. 1964b, n° 2; [3], p. 5) – and Φ^* is the Fourier transform of $\Phi\in L^2(G^2)$, defined as loc. cit.

The unitary operators in $L^2(G^2)$ given by (23a, b, c) are entirely analogous to those in $L^2(G)$ defined in Weil's paper (1964b, n° 13; [3], p. 18); in this sense the embedding (20) of $Sp(G)$ in $\mathbf{B}_0(G^2)$ may be regarded as natural. The analogy is related to a general property of Weil's theta representation (1964b, n° 20; [3], pp. 26–27)[2].

If G^* is isomorphic to G, (20) gives a realization of $Sp(G)$ as a group of unitary operators in $L^2(G^2)$ such that the Fourier transformation followed by an interchange of variables is an element of that group [cf. (23c)]; for $G \cong k^m$, k a local field, this is an analogue of Weil's unitary representation of the metaplectic group $Mp(k^m)$ in $L^2(k^m)$ (1964b, n° 34; [3], pp. 41–42). In this case, the matrices in $Sp(k^m)$ appearing on the left in (23a, b, c), with $\gamma = 1_m$, generate $Sp(k^m)$; this is contained in a well-known result of Witt (cf. [4], p. 335, Satz C, and the remark on p. 337 at the end of the Anhang, or [1], p. 326, Satz A 5.4). The formulae (23a, b, c) thus imply, in the case $G \cong k^m$, that the operators $\mathbf{r}^{(2)}(\sigma)$, $\sigma \in Sp(G)$, leave the closed linear subspace of all symmetric functions in $L^2(G^2)$ invariant; here f is called symmetric if f coincides almost everywhere on G^2 with some f' satisfying $f'(y,x) = f'(x,y)$ for all x, y in G. How far Witt's result extends to $Sp(G)$ for more general G is a question that needs investigation.

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[2] It is readily seen that, in the general case considered there, the following additional condition must be imposed on the isomorphism $\gamma: G^* \to G$. *For the restriction γ_* of γ to Γ_* the relation $|\gamma_*| = |\gamma|^{1/2}$ should hold,* where $|\gamma_*| dx_* = d\xi$ ($\xi = x_* \gamma_*$ and the Haar measure dx_* on Γ_* 'orthogonal' to $d\xi$ so that Poisson's formula holds). This condition is satisfied if Γ and Γ_* are both compact or both discrete and also (as may readily be shown) whenever γ is symmetric. In the case (23c) above the corresponding condition for the isomorphism $G^{*2} \to G^2: (x^*, y^*) \mapsto (y^*\gamma, x^*\gamma)$ obviously holds, for arbitrary $\gamma \in \mathrm{Is}(G^*, G)$ and Γ, Γ_* as in (8a, b); but for the isomorphism $(x^*, y^*) \mapsto (y^*\gamma_1, x^*\gamma_2)$ with γ_1, γ_2 in $\mathrm{Is}(G^*, G)$ it holds (as is easily seen) if and only if $|\gamma_1| = |\gamma_2|$. In this whole context it may be observed that, *if in the definition of $B_0(G, \Gamma)$* ([3], p. 25) *the additional condition is imposed that the restriction of σ to $\Gamma \times \Gamma_*$ should have Haar modulus* 1, *then Théorème* 4 ([3], p. 26) *holds as stated;* cf. A. Weil's remarks in [3], p. 445.